Differentiation C3

Specifications.

By the end of this unit you should be able to :

Use chain rule to find the derivative of composite functions.

Find the derivative of products and quotients.

Find the derivatives of trigonometric, logarithmic and exponential functions.

Work will also include turning points and the equations of tangents and normals.

The following differentials must be learnt.

Function	**Differential**
lnx	$\frac{1}{x}$
e^{mx}	me^{mx}
Sin t	Cos t
Cos t	-Sin t
Tan t	Sec^2 t
Sec t	Sec t Tan t
Cot t	$-Cosec^2$ t
Cosec t	-Cosec t cot t
Sin^n t	$nSin^{n-1}$ t Cos t
Cos^n t	$-nCos^{n-1}$ t Sin t
Sin (g(t))	g'(t)Cos (g(t))

Cos (g(t))	-g′(t)Sin (g(t))
Tan (g(t))	$g'(t)Sec^2$ (g(t))

Chain Rule

If y is a function of v and v, in turn, is a function of x, then:

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}$$

Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Composite Functions

$$y = (mx + c)^n$$

$$\frac{dy}{dx} = nm(mx + c)^{n-1}$$

$$y = (f(x))^n$$

$$\frac{dy}{dx} = nf'(x) \times (f(x))^{n-1}$$

The following example covers most of the ideas introduced above.

Example 1

Differentiate the following with respect to x simplify your answer as far as possible.

a) $(x + \ln 4x)^7$

b) $6 \sin^2 x + \sec 2x$

c) $x^{11} \tan 13x$

d) $x^5 e^{5x+5}$

e) $\dfrac{\cos 5x^5}{3x}$

a) $\dfrac{d}{dx}(x + \ln 4x)^7$

This is a composite function and chain rule must be used.

By chain rule

If $y = (x + \ln 4x)^7$ let $u = (x + \ln 4x)$

So $y = u^7$ $\frac{du}{dx} = 1 + \frac{1}{x}$ not $\frac{1}{4x}$

$\frac{dy}{du} = 7u^6$

So

$$\frac{d}{dx}(x + \ln 4x)^7 = 7u^6 \times \left(1 + \frac{1}{x}\right)$$

$$= 7(x + \ln 4x)^6 \left(1 + \frac{1}{x}\right)$$

b) The differential of sec2x is 2 sec 2x tan 2x (don't forget the 2's).

The $\sin^2 x$ requires substitution and the use of chain rule.

If $y = 6\sin^2 x$ let $u = \sin x$

then $y = 6u^2$ $\frac{du}{dx} = \cos x$

$\frac{dy}{du} = 12u$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \cos x \times 12u$$

$= 12\cos x \sin x = 6 \sin 2x$

Comment [u1]: Double angle formula
2SinxCosx = Sin2x

So finally:

$$\frac{d}{dx}(6\sin^2 x + \sec 2x) = 6 \sin 2x + 2 \sec 2x \tan 2x$$

c)

The rule for differentiating products is to differentiate the first and times it by the second and then add the differential of the second times by the first. Or in symbols:

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d}{dx}(x^{11} \tan 13x) = 11x^{10} \tan 13x + 13x^{11} \sec^2 13x$$

A lot of students forget to put the 13 back into the trigonometric function and write $\sec^2 x$.

d) $\frac{d}{dx}(x^5 \; e^{5x+5})$

Another product to differentiate, remember to differentiate the power on the e and bring it to the front.

$$\frac{d}{dx}(x^5\ e^{5x+5}) = 5x^4e^{5x+5} + 5x^5e^{5x+5}$$

$$= 5x^4e^{5x+5}(1+x)$$

e) $\frac{d}{dx}\left(\frac{\cos 5x^5}{3x}\right)$

We are asked to differentiate a quotient but this can be rewritten as a product. Most questions can be treated this way unless a question says specifically to use the quotient rule.

$$\frac{d}{dx}\left(\frac{\cos 5x^5}{3x}\right) = \frac{d}{dx}\left(\frac{x^{-1}\cos 5x^5}{3}\right)$$

Comment [u2]: Note that the 3 stays at the bottom.

Let u = $\frac{x^{-1}}{3}$ v = $\cos 5x^5$

Comment [u3]: Remember $\frac{d}{dx}(\cos(g(x))) = -g'(x)\sin(g(x))$

$\frac{du}{dx} = \frac{-x^{-2}}{3}$ $\frac{dv}{dx} = -25x^4 \sin 5x^5$

Therefore:

$$\frac{d\left(\frac{\cos 5x^5}{3x}\right)}{dx} = \frac{-x^{-2}}{3}\cos 5x^5 - \frac{x^{-1}}{3} \times -25x^4 \sin 5x^5$$

$$= -\frac{\cos 5x^5}{3x^2} + \frac{25x^3 \sin 5x^5}{3}$$

Example 2

Given that x = 8sin(7y + 3), find $\frac{dy}{dx}$.

The only thing that is new here is that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$x = 8\sin(7y + 3)$$

Don't forget to write the (7y + 3) in

$$\frac{dx}{dy} = 56\cos(7y + 3)$$

$$\frac{dy}{dx} = \frac{1}{56\cos(7y+3)} = \frac{\sec(7y+3)}{56}$$

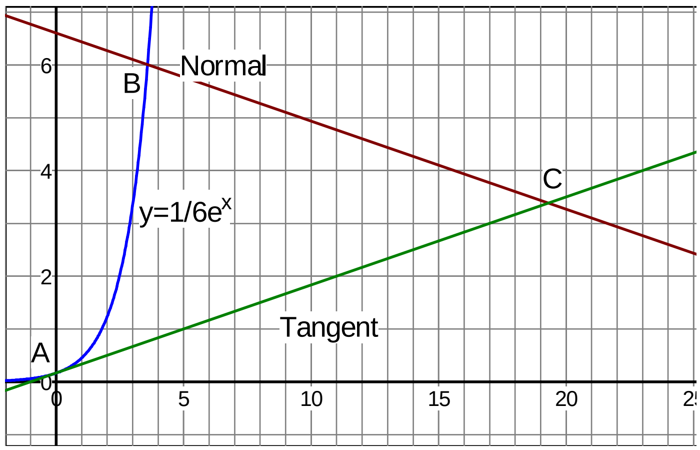
Equations of tangents and Normals

Example 3

The curve with equation $y = \frac{1}{6}e^x$ meets the y axis at the point A.

a) Prove that the tangent at A to the curve has equation 6y = x + 1

b) The point B has x-coordinate ln 36 and lies on the curve. The normal at B to the curve meets the tangent at A to the curve at the point C. Prove that the x coordinate of C is ln 6 + 17.5 and find the y coordinate of C.
It is always best practice to make a sketch of the diagram especially in this case where there are quite a few lines.



a) First find the gradient:

$$\frac{dy}{dx} = \frac{1}{6}e^x$$

At A, x = 0 therefore the gradient is $\frac{1}{6}$ as is the y coordinate.

$$y = mx + c$$

$$\frac{1}{6} = \frac{1}{6} \times 0 + c$$

Therefore

$$y = \frac{1}{6}x + \frac{1}{6}$$

and hence $6y = x + 1$

b) Find the equation of the normal at the point B with x coord of ln 36.

From part (a) the gradient function is $\frac{1}{6}e^x$ hence the gradient at B is 6 and so the normal gradient is $\frac{-1}{6}$. (y coord also 6)

$y = mx + c$

$6 = \dfrac{-\ln 36}{6} + c$

$y = \dfrac{-x}{6} + 6 + \dfrac{\ln 36}{6}$

The normal then meets the tangent from part (a) at the point *C*.

$\dfrac{1}{6}x + \dfrac{1}{6} = \dfrac{-x}{6} + 6 + \dfrac{\ln 36}{6}$

$2x = 35 + \ln 36$

$x = 17.5 + \dfrac{1}{2}\ln 36 = 17.5 + \ln 6$

Therefore the y coordinate can be found by substituting x = ln 6 + 17.5 into 6y = x + 1.

$6y = \ln 6 + 17.5 + 1$

$y = \dfrac{1}{6}\left(\ln 6 + \dfrac{37}{2}\right)$

Example 4

The curve c has equation $y = 4x^{\frac{7}{2}} - \ln 4x$, where x > 0. The tangent at the point *C* where x = 1 meets the x axis at the point *A*.

Prove that the x coordinate of A is $\frac{9+\ln 4}{13}$.

Once again start by differentiating to find the gradient when x = 1.

$$y = 4x^{\frac{7}{2}} - \ln 4x$$

$$\frac{dy}{dx} = 14x^{\frac{5}{2}} - \frac{1}{x}$$

When $x = 1$ grad = 13 y = 4 - ln4

y = mx + c

4 – ln4 = 13 + c

c = -9 – ln4 y = 13x – 9 – ln4

The line meets the x-axis at the point where y = 0. Therefore:

13x – 9 – ln4 = 0

$$x = \frac{9+\ln 4}{13}$$

Differentiating Quotients

Example 5

Given that $y = \frac{4x^2 - 16x + 7}{(x-2)^2}$, $x \neq 2$,

Show that $\frac{dy}{dx} = \frac{18}{(x-2)^3}$

It is obvious from the question that by using the quotient rule it will be easier to get the desired answer.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 4x^2 - 16x + 7$ $\qquad\qquad$ $v = (x - 2)^2$

$\frac{du}{dx} = 8x - 16$ $\qquad\qquad$ $\frac{dv}{dx} = 2(x-2)$

Therefore:

$$\frac{dy}{dx} = \frac{(x-2)^2(8x-16) - (4x^2 - 16x + 7) \times 2(x-2)}{(x-2)^4}$$

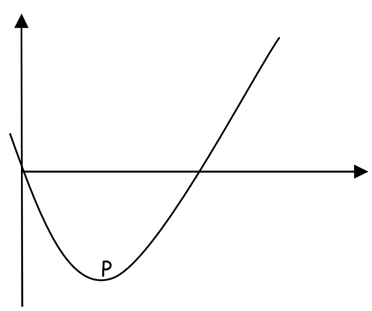
cancel a factor of $(x-2)$ top and bottom

$$= \frac{(x-2)(8x-16) - 8x^2 + 32x - 14}{(x-2)^3}$$

$$= \frac{8x^2 - 32x + 32 - 8x^2 + 32x - 14}{(x-2)^3}$$

$$= \frac{18}{(x-2)^3}$$

Example 6



The diagram shows part of the curve with equation,

$$y = (10x - 3) \tan 3x, \qquad 0 \le x < \frac{\pi}{4}$$

The curve has a minimum at the point P. The x coordinate of P is K. Show that K satisfies the equation $30K - 9 + 5 \sin 6K = 0$

As soon as you see the word minimum your first thought should be to differentiate and set it equal to zero.

Differentiating a product:

$$y = (10x - 3) \tan 3x$$

$$\frac{dy}{dx} = 10\tan 3x + 3(10x - 3)\sec^2 3x$$

At a minimum the gradient is zero

$$0 = 10\tan 3x + 3(10x - 3)\sec^2 3x$$

$$0 = 10\frac{\sin 3x}{\cos 3x} + \frac{3(10x - 3)}{\cos^2 3x}$$

Multiply by $\cos^2 3x$

$$0 = 10\sin 3x\cos 3x + 3(10x - 3)$$

$$0 = 5\sin 6x + 3(10x - 3)$$

If x = K

$$30K - 9 + 5\sin 6K = 0$$

A lot of the ideas outlined above are not complicated and the final example below deals with turning points and the differential of exponential functions.

Example 7

a) The curve, *C*, has equation

$$y = \frac{x}{9 + x^2}$$

Use calculus to find the coordinates of the tuning points of *C*.

b) Given that

$$y = (1 + e^{4x})^{\frac{5}{4}}$$

find the value of $\frac{dy}{dx}$ at the point x = $\frac{1}{4}\ln 3$.

a) The curve *C* is to be differentiated as a quotient

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

u = x   v = 9 + x^2

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{9 + x^2 - 2x^2}{(9 + x^2)}$$

$$\frac{dy}{dx} = \frac{9 - x^2}{(9 + x^2)}$$

Turning points exist where $\frac{dy}{dx}$ = 0 therefore the numerator must equal zero (why not the denominator?). The numerator is the difference of two squares and therefore the values of x must be +/- 3.
By substituting these values into y we get:

$$y = \frac{x}{9 + x^2} \qquad \left(3, \frac{1}{6}\right) \text{ and } \left(3, \frac{-1}{6}\right)$$

b) Given that

$$y = \left(1 + e^{4x}\right)^{\frac{5}{4}}$$

find the value of $\frac{dy}{dx}$ at the point x = $\frac{1}{4}\ln 3$.

This is a composite function. To differentiate it simply multiply by the power, multiply by the differential of the bracket and then multiply by the differential of the bracket.

$$y = \left(1 + e^{4x}\right)^{\frac{5}{4}}$$

$$\frac{dy}{dx} = \frac{5}{4} \times 4e^{4x} \times \left(1 + e^{4x}\right)^{\frac{1}{4}}$$

$$\frac{dy}{dx} = 5e^{4x}\left(1 + e^{4x}\right)^{\frac{1}{4}}$$

Let x = $\frac{1}{4}\ln 3$

$$\frac{dy}{dx} = 5e^{\ln 3}\left(1 + e^{\ln 3}\right)^{\frac{1}{4}}$$

$$\frac{dy}{dx} = 15\left(4^{\frac{1}{4}}\right)$$

C3 Differentiation is an area where marks can be scored easily. You should not find these questions difficult!